

**NUMERICAL SOLUTION
OF NONLINEAR PROBLEMS
OF SPATIAL EQUILIBRIUM FORMS
FOR THIN ELASTIC RODS**

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The problems of three-dimensional equilibrium forms of thin elastic rods are considered, in which one end of the rod is fixed and the other is loaded by a system of specified conservative forces. The method for constructing a sequence of approximate solutions with monotonically decreasing values of the energy functional is proposed. The method reported differs from that in [1] in that all angles determining the orientation of the main inertial axes of the rod cross sections are varied simultaneously.

1. Variational Statement of the Problem. The stable equilibrium form of a thin elastic rod with one end fixed and the other loaded by a system of given conservative forces \mathbf{P}_m ($m = 1, 2, \dots, M$) corresponds to the minimum of the functional

$$\Phi = \frac{1}{2} \int_0^l \sum_{k=1}^3 \{D_k(\omega_k - \omega_k^0)^2\} ds - \sum_{m=1}^M \mathbf{P}_m \cdot \mathbf{r}_m, \tag{1.1}$$

where l is the rod length; D_k ($k = 1, 2,$ and 3) are the flexural and torsional rigidities; ω_k, ω_k^0 ($k = 1, 2$ and 3) are the curvatures and torsion of the rod axis in the deformed and undeformed states

$$\begin{aligned} \omega_1 &= \frac{d\psi}{ds} \sin \vartheta \cdot \sin \varphi + \frac{d\vartheta}{ds} \cos \varphi, & \omega_2 &= \frac{d\psi}{ds} \sin \vartheta \cdot \cos \varphi - \frac{d\vartheta}{ds} \sin \varphi, \\ \omega_3 &= \frac{d\psi}{ds} \cos \vartheta + \frac{d\varphi}{ds}, & \omega_k^0 &= \omega_k \Big|_{\psi=\psi_0, \vartheta=\vartheta_0, \varphi=\varphi_0}; \end{aligned} \tag{1.2}$$

$\psi, \vartheta,$ and φ are the Euler angles determining the orientation of the basis vectors \mathbf{e}_i^l ($i = 1, 2, 3$) directed along the main inertial axes of the cross section and along the rod axis (Fig. 1); $\psi_0, \vartheta_0, \varphi_0$ are the Euler angles in the undeformed state; and \mathbf{r}_m ($m = 1, 2, \dots, M$) are the radius vectors of the points of application of the forces \mathbf{P}_m ($m = 1, 2, \dots, M$) in the equilibrium state.

It is assumed that the values and directions of the forces \mathbf{P}_m ($m = 1, 2, \dots, M$) are independent of the rod deformation, and hence the components P_m^i of these forces ($P_m^i = \mathbf{P}_m \cdot \mathbf{e}_i^i$) are constant in the Cartesian coordinates x_i ; with the basis vectors $\mathbf{e}^i = \mathbf{e}_i$.

We denote by \mathbf{r} the radius-vector of the end $s = l$ of the rod axis. Using the obvious equality

$$\frac{d\mathbf{r}}{ds} = \mathbf{e}_3^l$$

and assuming that

$$\mathbf{r}_m = \mathbf{r} + \mathbf{r}'_m, \quad \mathbf{r}'_m = (x'_k)_m \mathbf{e}_k^l,$$

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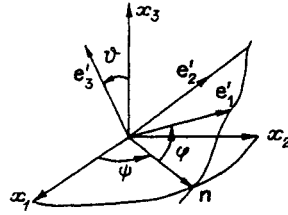


Fig. 1

we can write for the potential energy of the forces \mathbf{P}_m :

$$-\sum_{m=1}^M \mathbf{P}_m \cdot \mathbf{r}_m = -\left[\int_0^l \mathbf{F} \cdot \mathbf{e}'_3 ds + \mathbf{N}_k \cdot \mathbf{e}'_k \right], \quad (1.3)$$

where

$$\mathbf{F} = \sum_{m=1}^M \mathbf{P}_m; \quad \mathbf{N}_k = \sum_{m=1}^M P_m^i(x'_k)_m \mathbf{e}_i;$$

$$\mathbf{e}'_1 = \cos \varphi \cdot \mathbf{n} + \sin \varphi \cdot \mathbf{n}'; \quad \mathbf{e}'_2 = -\sin \varphi \cdot \mathbf{n} + \cos \varphi \cdot \mathbf{n}'; \quad \mathbf{e}'_3 = -\sin \vartheta \cdot \mathbf{n}'' + \cos \vartheta \cdot \mathbf{e}_3;$$

$$\mathbf{n} = \cos \psi \cdot \mathbf{e}_1 + \sin \psi \cdot \mathbf{e}_2; \quad \mathbf{n}' = \cos \vartheta \cdot \mathbf{n}'' + \sin \vartheta \cdot \mathbf{e}_3; \quad \mathbf{n}'' = -\sin \psi \cdot \mathbf{e}_1 + \cos \psi \cdot \mathbf{e}_2.$$

Functional (1.1) is defined on the functions ψ , ϑ , and φ , satisfying the conditions

$$\vartheta \Big|_{s=0} = 0, \quad (\psi + \varphi) \Big|_{s=0} = 0. \quad (1.4)$$

The coordinates x_k of the rod axis are related to ψ , ϑ , and φ by the formula

$$x_k = \int_0^s \mathbf{e}'_3 \cdot \mathbf{e}_k ds.$$

2. Approximation of the Energy Functional by a Quadratic Functional. Let $\tilde{\psi}$, $\tilde{\vartheta}$, and $\tilde{\varphi}$ be some functions satisfying conditions (1.4). By the algorithm proposed we construct the functions $\dot{\psi}(s)$, $\dot{\vartheta}(s)$, and $\dot{\varphi}(s)$ satisfying (1.4) such that

$$\Phi(\tilde{\psi} + \dot{\psi}, \tilde{\vartheta} + \dot{\vartheta}, \tilde{\varphi} + \dot{\varphi}) \leq \Phi(\tilde{\psi}, \tilde{\vartheta}, \tilde{\varphi}). \quad (2.1)$$

The equality in (2.1) is possible only for $\tilde{\psi}$, $\tilde{\vartheta}$, $\tilde{\varphi}$ corresponding to the extremum of the functional Φ (in this case $\dot{\psi} = \dot{\vartheta} = \dot{\varphi} = 0$).

We denote by Φ' the functional resulting from expansion of $\Phi(\tilde{\psi} + \dot{\psi}, \tilde{\vartheta} + \dot{\vartheta}, \tilde{\varphi} + \dot{\varphi})$ into a series with retention of summands that are linear and quadratic with respect to $\dot{\psi}$, $\dot{\vartheta}$, $\dot{\varphi}$:

$$\Phi' = \Phi_1 + \Phi_2,$$

$$\Phi_1 = \frac{1}{2} \int_0^l \left[\sum_{k=1}^3 \{D_k(\tilde{\omega}_k - \omega_k^0 + \dot{\omega}_k)^2\} - 2\mathbf{F} \cdot (\tilde{\mathbf{e}}'_3 + \dot{\mathbf{e}}'_3) \right] ds - \mathbf{N}_k \cdot (\tilde{\mathbf{e}}'_k + \dot{\mathbf{e}}'_k), \quad (2.2)$$

$$\Phi_2 = \frac{1}{2} \int_0^l \left[\sum_{k=1}^3 \{D_k(\tilde{\omega}_k - \omega_k^0)\dot{\omega}_k\} - \mathbf{F} \cdot \dot{\mathbf{e}}'_3 \right] ds - \frac{1}{2} \mathbf{N}_k \cdot \dot{\mathbf{e}}'_k.$$

Here

$$\dot{\omega}_1 = \frac{d\dot{\psi}}{ds} \sin \tilde{\vartheta} \cdot \sin \tilde{\varphi} + \frac{d\dot{\vartheta}}{ds} \cos \tilde{\varphi} + \tilde{\omega}_2 \dot{\varphi} + \dot{\vartheta} \frac{d\tilde{\psi}}{ds} \cos \tilde{\vartheta} \cdot \sin \tilde{\varphi};$$

$$\begin{aligned}
\omega_2 &= \frac{d\dot{\psi}}{ds} \sin \tilde{\vartheta} \cdot \cos \tilde{\varphi} - \frac{d\dot{\vartheta}}{ds} \sin \tilde{\varphi} - \tilde{\omega}_1 \dot{\varphi} + \dot{\vartheta} \frac{d\tilde{\psi}}{ds} \cos \tilde{\vartheta} \cdot \cos \tilde{\varphi}; \\
\omega_3 &= \frac{d\dot{\psi}}{ds} \cos \tilde{\vartheta} - \dot{\vartheta} \frac{d\tilde{\psi}}{ds} \sin \tilde{\vartheta} + \frac{d\dot{\varphi}}{ds}; \quad \dot{\mathbf{e}}'_k = \dot{\boldsymbol{\gamma}} \times \tilde{\mathbf{e}}'_k; \quad \dot{\boldsymbol{\gamma}} = \dot{\gamma}_k \tilde{\mathbf{e}}'_k; \\
\dot{\gamma}_1 &= \dot{\psi} \sin \tilde{\vartheta} \cdot \sin \tilde{\varphi} + \dot{\vartheta} \cos \tilde{\varphi}; \quad \dot{\gamma}_2 = \dot{\psi} \sin \tilde{\vartheta} \cdot \cos \tilde{\varphi} - \dot{\vartheta} \sin \tilde{\varphi}; \quad \dot{\gamma}_3 = \dot{\psi} \cos \tilde{\vartheta} + \dot{\varphi}; \\
\tilde{\omega}_k &= \left(\frac{d\tilde{\boldsymbol{\gamma}}}{ds} + \frac{d\dot{\boldsymbol{\gamma}}}{ds} \times \dot{\boldsymbol{\gamma}} \right) \cdot \tilde{\mathbf{e}}'_k; \quad \tilde{\mathbf{e}}'_k = \tilde{\boldsymbol{\gamma}} \times \tilde{\mathbf{e}}'_k + \dot{\boldsymbol{\gamma}} \times (\dot{\boldsymbol{\gamma}} \times \tilde{\mathbf{e}}'_k); \quad \tilde{\boldsymbol{\gamma}} = \tilde{\gamma}_k \tilde{\mathbf{e}}'_k; \quad \frac{d\tilde{\boldsymbol{\gamma}}}{ds} = \frac{d\tilde{\gamma}_k}{ds} \tilde{\mathbf{e}}'_k + \tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{\gamma}}; \\
\tilde{\boldsymbol{\omega}} &= \tilde{\omega}_k \tilde{\mathbf{e}}'_k; \quad \frac{d\dot{\boldsymbol{\gamma}}}{ds} = \dot{\omega}_k \tilde{\mathbf{e}}'_k; \quad \tilde{\gamma}_1 = \dot{\psi} \dot{\vartheta} \cos \tilde{\vartheta} \cdot \sin \tilde{\varphi} - \dot{\vartheta} \dot{\varphi} \sin \tilde{\varphi} + \dot{\psi} \dot{\varphi} \sin \tilde{\vartheta} \cdot \cos \tilde{\varphi}; \\
\tilde{\gamma}_2 &= \dot{\psi} \dot{\vartheta} \cos \tilde{\vartheta} \cdot \cos \tilde{\varphi} - \dot{\vartheta} \dot{\varphi} \cos \tilde{\varphi} - \dot{\psi} \dot{\varphi} \sin \tilde{\vartheta} \cdot \sin \tilde{\varphi}; \quad \tilde{\gamma}_3 = -\dot{\psi} \dot{\vartheta} \sin \tilde{\vartheta}.
\end{aligned}$$

The algorithm is based on the use of the obvious property: for sufficiently small a inequality (2.1) follows from the inequalities

$$\Phi' \leq \Phi(\tilde{\psi}, \tilde{\vartheta}, \tilde{\varphi}), \quad \max_s (|\dot{\psi}(s)|, |\dot{\vartheta}(s)|, |\dot{\varphi}(s)|) \leq a. \quad (2.3)$$

3. Difference Approximation. The rod axis is divided by the nodes $s_i = (i-1)/h$ ($i = 1, 2, \dots, N+1$, $h = l/N$) into N elements. The elements are enumerated by the numbers $i + 1/2$ ($i = 1, 2, \dots, N$), and the values corresponding to the element $i + 1/2$ bear the index $i + 1/2$. We assume that

$$\psi_{i+1/2} = \frac{1}{2}(\psi_i + \psi_{i+1}), \quad \left(\frac{d\psi}{ds} \right)_{i+1/2} = \frac{1}{h}(\psi_{i+1} - \psi_i)$$

(ψ_i are the values of ψ at the nodes). In a similar way, the quantities

$$\vartheta_{i+1/2}, \quad \varphi_{i+1/2}, \quad \left(\frac{d\vartheta}{ds} \right)_{i+1/2}, \quad \left(\frac{d\varphi}{ds} \right)_{i+1/2}$$

are defined in terms of ϑ_i and φ_i at the nodes. The sines and cosines in the element $i + 1/2$ are calculated from formulas of the form

$$(\sin \psi)_{i+1/2} = \sin \psi_{i+1/2}, \quad (\cos \psi)_{i+1/2} = \cos \psi_{i+1/2}.$$

Here and below, by Φ , Φ' , Φ_1 , and Φ_2 , we mean functionals obtained from difference approximation of the functions $\tilde{\psi}(s)$, $\tilde{\vartheta}(s)$, $\tilde{\varphi}(s)$, $\dot{\psi}(s)$, $\dot{\vartheta}(s)$, $\dot{\varphi}(s)$ and their derivatives in (1.1)–(1.3), (2.2), (2.3). Below conditions (1.4)

$$\dot{\vartheta}_1 = 0, \quad \dot{\psi}_1 + \dot{\varphi}_1 = 0 \quad (3.1)$$

are supplemented by the conditions

$$\dot{\psi}_1 = \dot{\psi}_2. \quad (3.2)$$

It is assumed that $\tilde{\psi}_1$, $\tilde{\vartheta}_1$, $\tilde{\varphi}_1$, $\tilde{\psi}_2$ satisfy the conditions

$$\tilde{\vartheta}_1 = 0, \quad \tilde{\psi}_1 + \tilde{\varphi}_1 = 0, \quad \tilde{\psi}_1 = \tilde{\psi}_2. \quad (3.3)$$

4. Minimization of Quadratic Functional. In the algorithm used herein for constructing the functions $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \dots, N + 1$) satisfying conditions (2.3), (3.1), and (3.2), the functional Φ'' is minimized:

$$\Phi'' = \Phi' + \Psi, \quad \Psi = \frac{1}{2} \sum_{i=2}^{N+1} \left[\varepsilon_i^{(1)} \dot{\psi}_i^2 + \varepsilon_i^{(2)} \dot{\vartheta}_i^2 + \varepsilon_i^{(3)} \dot{\varphi}_i^2 \right]$$

[$\varepsilon_i^{(s)}$ ($s = 1, 2, 3$) are nonnegative numbers]. It is obvious that the first inequality of (2.3) holds for those $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \dots, N + 1$) to which correspond the minimum of Φ'' . The introduction of the functional Ψ with numbers $\varepsilon_i^{(s)}$ ($s = 1, 2, 3$) is one of the means used below for the second inequality in (2.3) to hold.

Minimization of Φ'' involves two stages. In the first stage, we minimize the functional

$$\Phi^* = \Phi_1 + \Psi$$

over the class of functions $\dot{\psi}_i$, $\dot{\vartheta}_i$, and $\dot{\varphi}_i$ satisfying conditions (3.1) and (3.2). A solution of the system below corresponds to the minimum of Φ^* :

$$\begin{aligned} \mathbf{u}_1 = R_1 \mathbf{u}_2, \quad A_i \mathbf{u}_{i-1} + (B_i + E_i) \mathbf{u}_i + C_i \mathbf{u}_{i+1} = \mathbf{f}_i \quad (i = 2, 3, \dots, N), \\ A_{N+1} \mathbf{u}_N + (B_{N+1} + E_{N+1}) \mathbf{u}_{N+1} = \mathbf{f}_{N+1}, \end{aligned} \quad (4.1)$$

where \mathbf{u}_i are vectors with the components $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$. The matrices A_i , B_i , and C_i depend on $\tilde{\psi}_k$, $\tilde{\vartheta}_k$, and $\tilde{\varphi}_k$ ($k = i - 1, i, i + 1$); E_i is the diagonal matrix with the elements $\varepsilon_i^{(s)}$ ($s = 1, 2, 3$); and the vectors \mathbf{f}_i ($i = 2, \dots, N + 1$) are the discrepancies of the conditions of an extremum of the functional Φ .

The part of the functional Φ_1 quadratic for \mathbf{u}_i is nonnegative and becomes zero when

$$(\dot{\omega}_s)_{i+1/2} = 0, \quad s = 1, 2, 3, \quad i = 1, 2, \dots, N. \quad (4.2)$$

It follows from (4.2) that

$$\begin{aligned} \left[\frac{d\dot{\psi}}{ds} - \sin \tilde{\vartheta} \frac{d\tilde{\vartheta}}{ds} \dot{\varphi} + \cos \tilde{\vartheta} \frac{d\dot{\varphi}}{ds} \right]_{i+1/2} = 0, \quad \left[\frac{d\tilde{\psi}}{ds} \dot{\vartheta} - \cos \tilde{\vartheta} \frac{d\tilde{\vartheta}}{ds} \dot{\varphi} - \sin \tilde{\vartheta} \frac{d\dot{\varphi}}{ds} \right]_{i+1/2} = 0, \\ \left[\frac{d\dot{\vartheta}}{ds} + \frac{d\tilde{\psi}}{ds} \sin \tilde{\vartheta} \dot{\varphi} \right]_{i+1/2} = 0, \quad i = 1, 2, \dots, N. \end{aligned} \quad (4.3)$$

Using (3.1)–(3.3), we find that when

$$\tilde{\vartheta}_2 \neq 0, \quad \left\{ \left[1 + \left(\frac{h}{2} \frac{d\tilde{\psi}}{ds} \right)^2 \right] \sin \tilde{\vartheta} + \frac{h}{2} \frac{d\tilde{\vartheta}}{ds} \cos \tilde{\vartheta} \right\}_{i+1/2} \neq 0, \quad i = 2, \dots, N \quad (4.4)$$

Eqs. (4.3) are valid only in the case

$$\dot{\psi}_i = \dot{\vartheta}_i = \dot{\varphi}_i = 0, \quad i = 1, 2, \dots, N + 1.$$

It is further assumed that conditions (4.4) are fulfilled. Under these conditions a solution of the system (4.1) exists for any nonnegative numbers $\varepsilon_i^{(s)}$ ($s = 1, 2, 3$, $i = 1, 2, \dots, N + 1$) and becomes zero only in the case where the conditions of an extremum of the functional Φ hold:

$$\mathbf{f}_i = 0, \quad i = 2, \dots, N + 1.$$

The solution of the system (4.1) by the sweep method [2] can be interpreted as an element $\mathbf{u}_i^{(N)}$ of the sequence

$$\mathbf{u}_i^{(k)} = \mathbf{u}_i^{(k-1)} + \dot{\mathbf{u}}_i^{(k)}, \quad k = 1, 2, \dots, N, \quad i = 1, 2, \dots, N + 1,$$

where $\mathbf{u}_i^{(0)} = 0$ ($i = 1, 2, \dots, N + 1$); $\dot{\mathbf{u}}_i^{(k)}$ are determined sequentially for $k = 1, 2, \dots, N$ by the equations

$$\begin{aligned} \dot{\mathbf{u}}_1^{(k)} = R_1 \dot{\mathbf{u}}_2^{(k)}, \quad \frac{\partial}{\partial \dot{\mathbf{u}}_{in}^{(k)}} \left(\Phi^* \Big|_{\mathbf{u}_s = \mathbf{u}_s^{(k)}} \right) = 0, \quad i = 2, \dots, k + 1, \quad n = 1, 2, 3, \\ \dot{\mathbf{u}}_i^{(k)} = 0, \quad i = k + 2, \dots, N + 1. \end{aligned} \quad (4.5)$$

By $\dot{\mathbf{u}}_{in}^{(k)}$ we denote the components of the vector $\dot{\mathbf{u}}_i^{(k)}$. The quantities $\dot{\mathbf{u}}_i^{(k)}$ corresponding to the solution of Eqs. (4.5) are related to $\dot{\mathbf{u}}_{i+1}^{(k)}$ by the equality

$$\dot{\mathbf{u}}_i^{(k)} = R_i \dot{\mathbf{u}}_{i+1}^{(k)}, \quad i = 1, 2, \dots, k, \quad k = 1, 2, \dots, N,$$

in which the matrices R_i are independent of k . By using this property, the solution procedure can be reduced to the calculation of the vectors $\dot{\mathbf{u}}_{i+1}^{(i)}$ of the matrices R_i :

$$\begin{aligned} \dot{\mathbf{u}}_2^{(1)} &= G_2^{-1} \mathbf{f}_2, & G_2 &= A_2 R_1 + B_2 + E_2, & \dot{\mathbf{u}}_{i+1}^{(i)} &= G_{i+1}^{-1} (\mathbf{f}_{i+1} - A_{i+1} \dot{\mathbf{u}}_i^{(i-1)}), \\ G_{i+1} &= A_{i+1} R_i + B_{i+1} + E_{i+1}, & R_i &= -G_i^{-1} C_i, & i &= 2, 3, \dots, N, \end{aligned} \quad (4.6)$$

and also to the calculation of \mathbf{u}_i ($i = N+1, \dots, 1$) from the formulas

$$\mathbf{u}_{N+1} = \dot{\mathbf{u}}_{N+1}^{(N-1)}, \quad \mathbf{u}_i = \dot{\mathbf{u}}_i^{(i-1)} + R_i \mathbf{u}_{i+1}, \quad i = N, N-1, \dots, 2, \quad \mathbf{u}_1 = R_1 \mathbf{u}_2. \quad (4.7)$$

We denote

$$\dot{u}_{in}(0) = \dot{u}_{in}^{(i-1)} \Big|_{\varepsilon_i^{(s)}=0}, \quad \dot{u}_{in}(\varepsilon) = \dot{u}_{in}^{(i-1)} \Big|_{\varepsilon_i^{(s)}=\delta^{sn} \varepsilon_i^{(n)}}$$

(δ^{sn} is the Kronecker symbol). It follows from (4.6) that

$$\dot{u}_{in}(\varepsilon) = \frac{\dot{u}_{in}(0)}{1 + \varepsilon_i^{(n)} g_i^{nn}}, \quad \dot{u}_{in}(0) = g_i^{nk} v_{ik},$$

where g_i^{nk} is an element of the matrix $(A_i R_{i-1} + B_i)^{-1}$, and v_{ik} is the component of the vector $\mathbf{f}_{i-1} - A_i \dot{\mathbf{u}}_{i-1}^{(i-2)}$.

To determine the numbers $\varepsilon_i^{(s)}$, we used below the condition

$$|\dot{u}_{is}(\varepsilon)| < \Delta, \quad s = 1, 2, 3$$

(Δ is the specified number). In accordance with this condition, $\varepsilon_i^{(s)}$ are calculated by the formula

$$\varepsilon_i^{(s)} = \begin{cases} 0 & \text{at } |\dot{u}_{is}(0)| \leq \Delta, \\ \frac{\dot{u}_{is}(0)}{\Delta |g_i^{ss}|} - \frac{1}{g_i^{ss}} & \text{at } |\dot{u}_{is}(0)| > \Delta. \end{cases} \quad (4.8)$$

The introduction of Δ is one of the ways of controlling the values $\psi_i, \vartheta_i, \varphi_i$ ($i = 1, 2, \dots, N+1$) for which the functional Φ^* has a maximum. It follows from (4.6)–(4.8) that these functions tend to zero as $\Delta \rightarrow 0$. In the calculations the authors usually assumed that $\Delta = 0.1$.

The second stage of minimizing the functional Φ'' consists in correcting \mathbf{u}_i ($i = 1, 2, \dots, N+1$) corresponding to a minimum of Φ^* . If for these values the functional Φ_2 is less than or equal to zero, the first of the inequalities (2.3) holds and no correction is made. If Φ_2 is greater than zero, then \mathbf{u}_i ($i = 1, 2, \dots, N+1$) corresponding to the minimum of Φ^* are multiplied by a correcting multiplier α whose value is defined by the minimum condition of Φ'' .

The functional Φ^* is the sum of three summands:

$$\Phi^* = \tilde{\Phi} + L + Q.$$

Here $\tilde{\Phi} = \tilde{\Phi}(\tilde{\psi}, \tilde{\vartheta}, \tilde{\varphi})$; L and Q are the first- and second-power functions homogeneous with respect to \mathbf{u}_i . The minimum of Φ^* is achieved when

$$L + 2Q = 0 \quad (4.9)$$

(which can be obtained, for example, from the equality to zero of the variation of the functional Φ^* after replacing in it the variations of $\delta \mathbf{u}_i$ by \mathbf{u}_i). With the help of (4.9) the condition of minimum Φ'' with respect to α is written as

$$\alpha = Q / (Q + \Phi_2). \quad (4.10)$$

The values of the functionals Φ'' , Q , and Φ_2 corresponding to those obtained as a result of correction

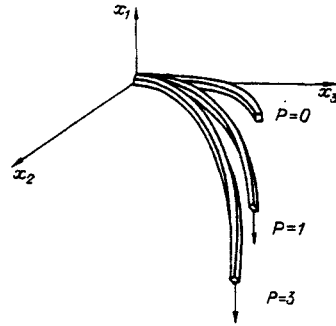


Fig. 2

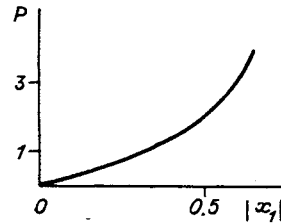


Fig. 3

of $\dot{\psi}_i$, $\dot{\vartheta}_i$, and $\dot{\varphi}_i$ are related by the equality

$$\Phi'' = \tilde{\Phi} - (Q + \Phi_2).$$

Thus, using the formulas (4.6) and (4.7) with $\varepsilon_i^{(s)}$ in accordance with (4.8) and introducing, if necessary, the correcting multiplier α following (4.10), one can obtain the values $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \dots, N+1$) at which

$$\Phi' \leq \Phi'' = \tilde{\Phi} - (Q + |\Phi_2|). \quad (4.11)$$

The values $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \dots, N+1$) can be controlled by the number Δ in (4.8).

5. Minimization of the Energy Functional. If, with the values $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$ ($i = 1, 2, \dots, N+1$) obtained from the minimization of the functional Φ'' , inequality (2.1) holds, the functions $\tilde{\psi}_i$, $\tilde{\vartheta}_i$, $\tilde{\varphi}_i$ are replaced by $\tilde{\psi}_i + \dot{\psi}_i$, $\tilde{\vartheta}_i + \dot{\vartheta}_i$, $\tilde{\varphi}_i + \dot{\varphi}_i$ and the following approximation is calculated.

If inequality (2.1) fails, the sequence of correcting multipliers γ^m ($m = 1, 2, 3, \dots$), $0 < \gamma < 1$ is introduced. In the calculations the authors usually assumed that $\gamma = 1/2$. In replacing $\dot{\psi}_i$, $\dot{\vartheta}_i$, $\dot{\varphi}_i$ by $\gamma^m \dot{\psi}_i$, $\gamma^m \dot{\vartheta}_i$, $\gamma^m \dot{\varphi}_i$ the value of the functional Φ'' increases but remains smaller than $\tilde{\Phi}$:

$$\begin{aligned} \Phi'' &= \tilde{\Phi} - \gamma^m(2 - \gamma^m)Q + \gamma^{2m}\Phi_2 & \text{at } \Phi_2 \leq 0, \\ \Phi'' &= \tilde{\Phi} - \gamma^m(2 - \gamma^m)(Q + \Phi_2) & \text{at } \Phi_2 > 0 \end{aligned} \quad (5.1)$$

[the values of Q and Φ_2 in (5.1) are the same as in (4.11)]. Since Φ'' remains smaller than $\tilde{\Phi}$ for all m , there is an m such that inequality (2.1) holds.

Thus, if $\tilde{\psi}_i$, $\tilde{\vartheta}_i$, $\tilde{\varphi}_i$ ($i = 1, 2, \dots, N+1$) do not satisfy the conditions of an extremum of the functional Φ , one can construct a new approximation with the lower value of the energy functional by means of the above method.

6. Flexure and Torsion of a Bar with a Curvilinear Axis as a Ring Quarter. Figure 2 shows the forms of equilibrium of a square bar with a curvilinear axis as a ring quarter ($\psi_0 = \varphi_0 = 0$, $\vartheta_0 = -(\pi/2)s$) under flexure by the force $\mathbf{P} = -P\mathbf{e}_1$ applied at the end $s = 1$ of the bar axis. Here and below P is a dimensionless quantity equal to the force multiplied by l^2/D_2 ; and s , x'_i , and x_i are the ratios of the arc length and coordinates to the bar axis l .

As a first approximation, we took $\psi = \psi_0$, $\vartheta = \vartheta_0$, $\varphi = \varphi_0$. Ten and twenty iterations were required to obtain a solution at $P = 1$ and 3, respectively. Here and below as a criterion of cessation of the process we took the condition

$$\max_i |f_i| < 10^{-3}.$$

Figure 3 shows the force P as a function of the coordinate x_1 at the end $s = 1$ of the bar axis.

7. Flexure and Torsion of a Rectilinear Bar by a Transverse Force Applied to Its End. We calculated the equilibrium states of a bar under flexure and torsion by a force $\mathbf{P} = P\mathbf{e}_2$ applied to the bar

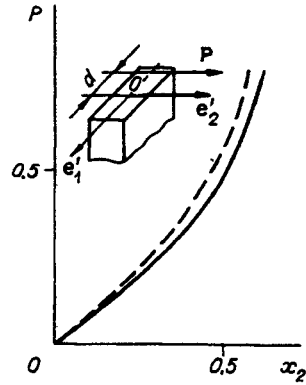


Fig. 4

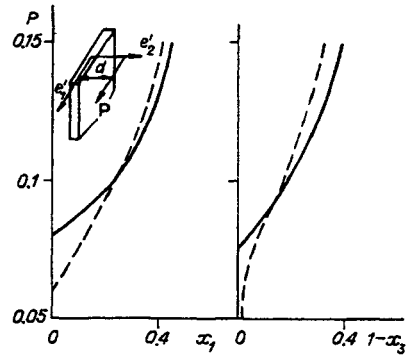


Fig. 5

end at the point $x'_1 = -d$, $x'_2 = 0$ (Fig. 4). We considered the case where

$$D_1/D_2 = 1/4, \quad D_3/D_2 = 0.05. \quad (7.1)$$

The solid and dashed curves in Fig. 4 show the dependence of the coordinate x_2 of the end $s = 1$ of the bar axis on P at $d = 0$ and 0.1 respectively.

It follows from the calculations of the equilibrium forms with torsional rigidities different from those in (7.1), that as D_3 increases, the difference between the shapes of the bar axis corresponding to $d = 0$ and 0.1 decreases and practically vanishes for $D_3/D_1 = 1$. The calculations performed made it possible to conclude that in the example cited, flexure is the principal factor determining the shape of the bar axis, whereas torsion has little effect on the bar axis.

We also considered the equilibrium states of a band of rectangular cross section with ratio of the sides $1 : 8$ under flexure and torsion by a transverse force $\mathbf{P} = P\mathbf{e}_1$ applied to the point $x'_1 = 0$, $x'_2 = d$ of the band end (Fig. 5). In this case, the difference between the shape of the bar axis corresponding to $d = 0$ and 0.1 becomes significant and is shown in Fig. 5, where the solid and dashed curves represent the dependences of the coordinates x_1 and x_3 of the end $s = 1$ of the bar axis on P at $d = 0$ and 0.1 .

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